Options Implied Probability Distributions (OIPD)

- extracts risk-neutral probability distribution of an assets price at a specific future date from options prices

- outputs pdf and cdf of random variable S_t the price of the option @ expiration.

This is:

- A pricing measure
- consistent with no arbitrage
- Fully implied by traded options

This is NOT:

- a forecast
- a real-world probability
- belief distribution

Breeden - Litzenberger:

Given Pricing function $C(K)$

the risk neutral probability density function $f(K) = e^{rT} \frac{d^2 C(K)}{dK^2}$

Example: lets say that the value of a European call option is

$$C(K,T) = e^{-r(T-t)} \mathbb{E}[\max(S_T - K, 0)]$$

where T is expiry time

S_T is price @ expiry

which is also represented as

pdf on random variable S_T

$$C(K,T) = e^{-r(T-t)} \int_{-\infty}^{\infty} \max(S_T - K, 0) f(S_T) \, dS_T$$

Now say we have a butterfly spread with a payout of ΔK. say it pays with probability p.

then, a reasonable non risk-seeking nor risk averse person would pay $\Delta K p$ for this option (remember expected value of Bernoulli distribution is p)

So if X is the amount we will pay, then $X = p\Delta K$

$$\Rightarrow p = \frac{X}{\Delta K}$$

strike	midline price
251	1.46
250	1.21
249	1.16

Butterfly spread buys 251, sells 2x250, buys 249

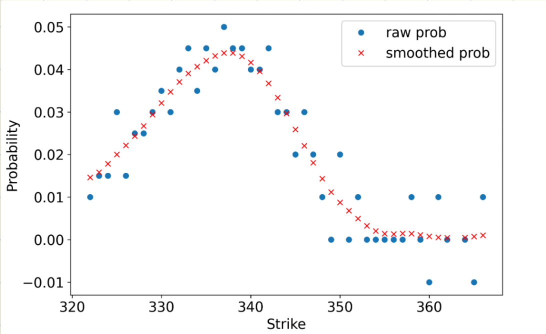
$\Rightarrow X = 1.46 - 2(1.21) + 1.16 = 0.2$

$\Rightarrow p = \frac{X}{\Delta K} = \frac{0.2}{1} = 0.2 = 20\%$ chance of profit

Repeat for every possible butterfly

↓ Smooth with Gaussian filter

↓ fit spline.



Moving from Discrete to Continuous Probabilities

Our old formula $p = \frac{X}{\Delta K}$ $f(\Delta K) = \frac{X}{\Delta K}$ is a pmf.

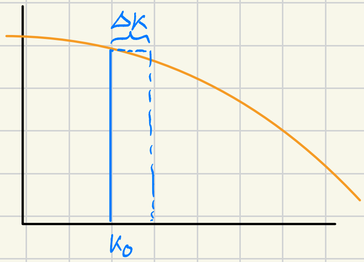
we want a pdf. recall the differences:

PMF	PDF
$\sum_{\Delta K} f(\Delta K) = 1$	$\int_{-\infty}^{\infty} f(\Delta K) = 1$
$P(\Delta K) = f(\Delta K)$	$P(A \leq \Delta K \leq B) = \int_A^B f(\Delta K)$

So, using a PDF, the probability the price at expiry S_T is within a band of width ΔK is

$$P(K_0 \leq S_T \leq K_0 + \Delta K) = \int_{K_0}^{K_0+\Delta K} f(S_T)\, dS_T$$

$$\approx f(K_0)\Delta K$$



As $\Delta K \to 0$: $P(K_0 \leq S_T \leq K_0 + \Delta K)$ becomes $P(K_0 = S_T)$

$$P(S_T) = \lim_{\Delta K \to 0} f(S_T)\Delta K$$

$$\Rightarrow f(S_T) = \lim_{\Delta K \to 0} \frac{P(S_T)}{\Delta K}$$

$$f(S_T) = \lim_{\Delta K \to 0}\left(\frac{X/\Delta K}{\Delta K}\right) = \lim_{\Delta K \to 0}\left(\frac{X}{\Delta K^2}\right)$$

using butterfly formula $\longrightarrow$ $= \lim_{\Delta K \to 0}\left(\frac{C(S_T+\Delta K) - 2C(S_T) + C(S_T - \Delta K)}{\Delta K^2}\right)$

$= \frac{d^2C}{dK^2}$ Plugging in $C(K,T) = e^{-rT}\mathbb{E}[\max(S_T - K, 0)]$

$$= e^{-rT}\int_K^{\infty}(x-K)f(x)\,dx$$

We derived the Breeden-Litzenberger formula: $\boxed{f(K) = \frac{d^2C}{dK^2}e^{-rT}}$

it may seem that this result is only applicable if everyone only buys butterfly spreads. Consider an alternative derivation:

$$C(K,T) = e^{-rT}\mathbb{E}[\max(S_T - K, 0)]$$

$$= e^{-rT}\int_0^{\infty}\max(S_T - K, 0) f(S_T)\,dS_T$$

$$= e^{-rT}\int_K^{\infty}(x-K)f(x)\,dx$$

$$= e^{rT}\left(\int_K^{\infty} x f(x)\,dx - K\int_K^{\infty} f(x)\,dx\right)$$

$$\frac{dC}{dK} = e^{-rT}\left(\frac{d}{dK}\int_K^{\infty} x f(x)\,dx - \frac{d}{dK}K\int_K^{\infty} f(x)\,dx\right)$$

$$= e^{-rT}\left(-Kf(K) - \left(\int_K^{\infty} f(x)\,dx - Kf(K)\right)\right)$$

$$= e^{-rT}\left(\int_K^{\infty} f(x)\,dx\right)$$ since f is a PDF:

$$= e^{-rT}\left(\underbrace{\int_{-\infty}^{K} f(x)\,dx - 1}_{\text{CDF!}}\right)$$ Derivative of CDF = PDF

$$\frac{d^2C}{dK^2} = \frac{d}{dK}\left(e^{-rT}\int_{-\infty}^{K} f(x)\,dx\right)$$

$$= e^{-rT} f(x)$$

$$\Rightarrow f(x) = \frac{d^2C}{dK^2}e^{rT}$$